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The extended phase space: a formalism for the study of quantum noise and quantum correlations

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Abstract. Extended Wigner and extended Weyl functions which are quartic functions of the wavefunction are introduced and their properties are studied. They provide an insight into the connection between quantum noise and quantum correlations. The general theory is applied to the examples of coherent states and thermal states. The construction of the extended Wigner function from quantum tomography experiments is also discussed and exemplified for the cases of superposition and also for a mixture of two coherent states.

1. Introduction

Quantum mechanically, a particle is described by its wavefunction in the x-representation $s(x)$ or by its wavefunction in the p-representation $\tilde{s}(p)$. These two functions are related to each other through a Fourier transform. Quantum phase space methods [1–3] have introduced functions in the x–p plane for a description of the particle in phase space, which is consistent with quantum mechanics. The Wigner function $W(x, p)$, where x and p are position and momentum is the most popular one, and methods have been developed recently for its measurement (Wigner tomography [4]). The Weyl function $\tilde{W}(X, P)$ where X and P are position and momentum *increments* is also another such function. We have recently studied its properties in [5] and interpreted it as a generalized correlation function. Its measurement using electron interferometry has been proposed in [6]. The Wigner function is related to the Weyl function through a two-dimensional Fourier transform and using this we have introduced, in [7], an extended phase space x–p–X–P (position–momentum–position increment–momentum increment) and proved uncertainty relations for $\delta X \delta p$ and $\delta P \delta x$.

In this paper we elaborate this idea further by introducing the extended Wigner function $W_e(x, p, X, P)$ and the extended Weyl function $\tilde{W}_e\,(x', p', X', P')$. The quantum phase space formalism is based heavily on the Fourier transform between the $s(x)$ and $\tilde{s}(p)$; and using it we can prove most of the properties of the Wigner and Weyl functions. The extended phase space formalism originates from the fact that the Wigner function $W(x, p)$ and Weyl function $\tilde{W}(X, P)$ are related through a two-dimensional Fourier transform and using it we have proved uncertainty relations for the dual variables (x, P) and also (p, X). The physical significance of these uncertainty relations is that they provide a deeper insight into the relation between quantum noise (described by δx, δp) and quantum correlations (described by δX, δP). Here we go one step further, and introduce the extended Wigner and extended Weyl functions in $x - p - X - P$ and study their properties. These functions simultaneously show both quantum noise and quantum correlations.

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The Wigner and Weyl functions involve products of the wavefunction at two different points of phase space: $(x - \frac{1}{2}X, p - \frac{1}{2}P)$ and $(x + \frac{1}{2}X, p + \frac{1}{2}P)$. Integration over X, P gives the Wigner function; and integration over x, p gives the Weyl function. They are both *quadratic* functions of the wavefunction and, consequently, the superposition of two states $s_1(x) + s_2(x)$ gives cross-terms which quantify the interference. The Weyl function shows the correlations in the system because it involves the integral of the wavefunction with the shifted (in both position and momentum) wavefunction.

The extended Wigner and extended Weyl functions involve products of the wavefunction at four different points of phase space. They are both *quartic* functions of the wavefunction and consequently, the superposition of two states $s_1 + s_2$ gives cross-terms of the type $s_1^3 s_2$, $s_1^2 s_2^2$ and $s_1 s_2^3$. Note that cross-terms such as $s_1^2 s_2^2$ involve the product of two probabilities and even exist in the case of mixed states.

In section 2 we define our notation and introduce the Wigner and Weyl functions and the uncertainty relations for $\delta X \delta p$ and $\delta P \delta x$. In section 3 we present the extended Wigner and extended Weyl functions and study their properties. In section 4 we present examples which elucidate the physical meaning of the extended Wigner and extended Weyl functions. In section 5 we discuss the construction of the extended Wigner function from quantum tomography experiments. We conclude in section 6 with a discussion of our results.

2. Uncertainty relations in an extended phase space

We consider the harmonic oscillator Hilbert space H spanned by the number eigenstates $\{|N\rangle; N = 0, 1, 2 \ldots\}$. We also consider the coherent states

$$|A\rangle = \exp\left(-\frac{|A|^2}{2}\right) \sum_{N=0}^{\infty} \frac{A^N}{(N!)^{1/2}} |N\rangle = D(A)|0\rangle \tag{1}$$

$$D(A) = \exp[Aa^\dagger - A^* a] \tag{2}$$

where $a^\dagger$, a are the usual creation and annihilation operators. All our quantities are in units in which $\hbar = c = k_B = 1$. $D(A)$ is the displacement operator which can also be expressed in terms of the position and momentum operators $\hat{x}$, $\hat{p}$ as

$$D(x, p) \equiv D\left(A = \frac{x + \mathrm{i}p}{\sqrt{2}}\right) = \exp(\mathrm{i}p\hat{x} - \mathrm{i}x\hat{p}). \tag{3}$$

The Wigner function of a state described by a density matrix ρ is defined as

$$\begin{aligned} W(x, p) &= \frac{1}{2\pi} \int \mathrm{d}X \langle x + \tfrac{1}{2}X|\rho|x - \tfrac{1}{2}X\rangle \exp(-\mathrm{i}Xp) \\ &= \frac{1}{2\pi} \int \mathrm{d}P \langle p + \tfrac{1}{2}P|\rho|p - \tfrac{1}{2}P\rangle \exp(\mathrm{i}Px). \end{aligned} \tag{4}$$

The equivalence between these expressions is known in the literature [1–3].

Another function which is useful in phase space methods is the Weyl function which is defined as

$$\begin{aligned} \tilde{W}(X, P) &= \int \mathrm{d}x \langle x + \tfrac{1}{2}X|\rho|x - \tfrac{1}{2}X\rangle \exp(-\mathrm{i}Px) \\ &= \int \mathrm{d}p \langle p + \tfrac{1}{2}P|\rho|p - \tfrac{1}{2}P\rangle \exp(\mathrm{i}pX) = \mathrm{Tr}[\rho D(X, P)]. \end{aligned} \tag{5}$$

It is known that these expressions are equal [1–3]. We have shown in [5] that the Weyl function can be understood as a *generalized correlation function*, where X and P are position and momentum *increments*. Correlation functions are the overlaps of the shifted wavefunction $s(x+X)$ with the wavefunction $s(x)$. Here we shift the wavefunction in both position and momentum and in this sense the Weyl function is a generalized correlation function.

For later purposes we use the expansion

$$D(X,P) = \sum_{N=0}^{\infty} \frac{(\mathrm{i}P\hat{x} - \mathrm{i}X\hat{p})^N}{N!} \tag{6}$$

to prove that the Weyl function can be expressed as

$$\tilde{W}(X,P) = \sum_{m,n} \frac{(\mathrm{i}P)^m(-\mathrm{i}X)^n}{m!n!}\sigma_{mn} \tag{7}$$

where σ_{mn} are the correlations

$$\sigma_{mn} = \mathrm{Tr}[\rho\{\hat{x}^m\hat{p}^n\}_{Weyl}]. \tag{8}$$

The subscript Weyl indicates symmetrized products of the non-commuting operators.

The Wigner function is related to the Weyl function through the Fourier transform [5, 7]:

$$\tilde{W}(X,P) = \iint \mathrm{d}x\,\mathrm{d}p\, W(x,p)\exp[-\mathrm{i}(Px - pX)] \tag{9}$$

where (x, P) and (X, p) are dual variables. Note also that [5, 7]

$$\frac{1}{2\pi}\iint |\tilde{W}(X,P)|^2\,\mathrm{d}X\,\mathrm{d}P = 2\pi\iint [W(x,p)]^2\,\mathrm{d}x\,\mathrm{d}p = \mathrm{Tr}[\rho^2] \leqslant 1. \tag{10}$$

Equations (9), (10) show that we can treat the Wigner and Weyl functions as two-dimensional 'wavefunctions', with x, p as 'positions' and X, P as 'momenta'. We stress that there are several limitations associated with this line of thinking. The operators x, p do not commute; the Wigner function is always real (not complex); and an arbitrary (normalizable) two-dimensional function is not necessarily a Wigner function (it is a deep problem to find constraints which a function should obey in order to be a Wigner function [8]). Nevertheless, relations (9), (10) motivate us to treat $W(x,p)$ and $\tilde{W}(X,P)$ as dual wavefunctions and to introduce an extended phase space x–p–X–P.

In [7] we studied the first steps of such a formalism. We introduced the uncertainties δX,δP and δx,δp in terms of the *squares* of the Weyl and Wigner functions, correspondingly. The quantities δX, δP provide a measure for the correlations in the quantum state ρ. Indeed, in [7] we explained that the Weyl function can be interpreted as a generalized correlation function and we gave several mathematical relations that led to the interpretation that $|\tilde{W}(X,P)|^2$ is a probability density for the correlation function. Therefore, the widths δX, δP which are associated with $|\tilde{W}(X,P)|^2$ quantify the correlations in the quantum state ρ. The quantities δx, δp describe quantum noise, such as the more conventional uncertainties Δx, Δp (which are defined in terms of the *first* power of the Wigner function). We have introduced them because they play a dual role to δX, δP in the sense that they obey uncertainty relations presented in [7], and also given below.

For pure states the δx, δp, δX, δP are related to the standard uncertainties Δx and Δp [7]:

$$\delta x = 2^{-1/2}\Delta X \qquad \delta p = 2^{-1/2}\Delta P \qquad \delta X = 2^{1/2}\Delta x \qquad \delta P = 2^{1/2}\Delta p \tag{11}$$

but for mixed states they are different. We have shown the existence of the uncertainty relations among the dual variables:

$$\delta X\delta p \geqslant \tfrac{1}{2}\,\mathrm{Tr}[\rho^2] \qquad \delta x\delta P \geqslant \tfrac{1}{2}\,\mathrm{Tr}[\rho^2] \tag{12}$$

which provide a deeper insight into the connection between quantum noise (as quantified by δx, δp) and quantum correlations (as quantified by δX, δP). For pure states these uncertainty relations reduce to the usual uncertainty relations; but for mixed states they are different. Uncertainty relations involving the usual uncertainties Δx and Δp for mixed states, have been studied in depth in [9]. Our uncertainty relations of equation (12) confirm our ideas about an extended phase space x–p–X–P, where the variables (x, P) and also (p, X) are dual to each other. In this extended phase space we have 'wavefunctions' of two variables (one from each pair). Above, we have used the $W(x, p)$ and its two-dimensional Fourier transform $\tilde{W}(X, P)$. However, we can also define the $W'(x, X)$ and $\tilde{W}'(p, P)$ as follows:

$$\begin{aligned} W'(x, X) &= \int \mathrm{d}p\, W(x, p) \exp(\mathrm{i}pX) \\ \tilde{W}'(p, P) &= \int \mathrm{d}x\, W(x, p) \exp(-\mathrm{i}Px) \\ \tilde{W}'(p, P) &= \frac{1}{2\pi} \iint \mathrm{d}x\, \mathrm{d}X\, W'(x, X) \exp[-\mathrm{i}(Px - pX)]. \end{aligned} \tag{13}$$

These ideas lead us naturally to the extended Wigner and extended Weyl functions, which are discussed below.

3. Extended Wigner and extended Weyl functions

The extended Wigner function is defined as

$$\begin{aligned} W_e(x, p, X, P) &= (2\pi)^2 \iint \mathrm{d}x'\, \mathrm{d}p'\, W(x + \tfrac{1}{2}x', p + \tfrac{1}{2}p') W(x - \tfrac{1}{2}x', p - \tfrac{1}{2}p') \\ &\quad \times \exp[\mathrm{i}(Xp' - Px')] \\ &= \iint \mathrm{d}X'\, \mathrm{d}P' \tilde{W}^*(X + \tfrac{1}{2}X', P + \tfrac{1}{2}P') \tilde{W}(X - \tfrac{1}{2}X', P - \tfrac{1}{2}P') \\ &\quad \times \exp[\mathrm{i}(X'p - P'x)] \end{aligned} \tag{14}$$

where the subnscript 'e' indicates 'extended'. It is easily seen that the extended Wigner function is a real function. Using the fact that $\tilde{W}(X, P) = \tilde{W}^*(-X, -P)$ we prove that $W_e(x, p, X, P) = W_e(x, p, -X, -P)$.

The extended Weyl function is defined as

$$\begin{aligned} \tilde{W}_e(x', p', X', P') &= (2\pi)^2 \iint \mathrm{d}x\, \mathrm{d}p\, W(x + \tfrac{1}{2}x', p + \tfrac{1}{2}p') W(x - \tfrac{1}{2}x', p - \tfrac{1}{2}p') \\ &\quad \times \exp[-\mathrm{i}(X'p - P'x)] \\ &= \iint \mathrm{d}X\, \mathrm{d}P\, \tilde{W}^*(X + \tfrac{1}{2}X', P + \tfrac{1}{2}P') \tilde{W}(X - \tfrac{1}{2}X', P - \tfrac{1}{2}P') \\ &\quad \times \exp[\mathrm{i}(Xp' - Px')]. \end{aligned} \tag{15}$$

In order to prove the equalities in equations (14), (15) we substitute equation (9) into these equations, perform the integration and obtain delta functions, which with another integration give the desired result. Because the Wigner function is always real, no complex conjugate is required in equations (14), (15).

The extended Weyl function is related to the extended Wigner function through the four-dimensional Fourier transform

$$\begin{aligned} \tilde{W}_e(x', p', X', P') &= \frac{1}{4\pi^2} \int \mathrm{d}x\, \mathrm{d}p\, \mathrm{d}X\, \mathrm{d}P\, W_e(x, p, X, P) \\ &\quad \times \exp\{-\mathrm{i}[(pX' - xP') - (Xp' - Px')]\}. \end{aligned} \tag{16}$$

This is proved if we substitute $W_e(x, p, X, P)$ in the right-hand side of equation (16) with equation (14). We get delta functions which after integration give the expressions of equation (15) which are $\tilde{W}_e(x', p', X', P')$.

Inserting equation (7) into (14), (15) we express the extended Wigner and extended Weyl functions in terms of the correlations σ_{mn} of equation (8):

$$W_e(x, p, X, P) = \sum_{m,n,k,l} W_e(m, n, k, l; x, p, X, P)\sigma^*_{mn}\sigma_{kl} \tag{17}$$

$$\tilde{W}_e(x', p', X', P') = \sum_{m,n,k,l} \tilde{W}_e(m, n, k, l; x', p', X', P')\sigma^*_{mn}\sigma_{kl} \tag{18}$$

where

$$W_e(m, n, k, l; x, p, X, P) = \iint \mathrm{d}X'\,\mathrm{d}P'\,(i)^{m-n-k+l}\exp[\mathrm{i}(X'p - P'x)] \times \frac{(X + \frac{1}{2}X')^m}{m!}\frac{(P + \frac{1}{2}P')^n}{n!}\frac{(X - \frac{1}{2}X')^k}{k!}\frac{(P - \frac{1}{2}P')^l}{l!} \tag{19}$$

and

$$\tilde{W}_e(m, n, k, l; x', p', X', P') = \iint \mathrm{d}X\,\mathrm{d}P\,(i)^{-m-n+k+l}\exp[\mathrm{i}(Xp' - Px')] \times \frac{(X + \frac{1}{2}X')^m}{m!}\frac{(P + \frac{1}{2}P')^n}{n!}\frac{(X - \frac{1}{2}X')^k}{k!}\frac{(P - \frac{1}{2}P')^l}{l!}. \tag{20}$$

Note that the Wigner and the Weyl functions are linear functions of σ_{mn}; in contrast to the extended Wigner and extended Weyl functions which are quadratic functions of σ_{mn}. In principle, knowledge of *all* σ_{mn} gives the Weyl function (equation (7)) and the density matrix. Therefore, quantities such as δx, δp, δX, δP and the $W_e(x, p, X, P)$, $\tilde{W}_e(x', p', X', P')$ are, *in principle,* functions of σ_{mn}. In practice, these quantities contain additional information about the system, in comparison with the σ_{mn}. Firstly, because in practice only the first few σ_{mn} will be measured; and secondly, because there are difficulties with convergence and it is not always easy to find explicit expressions that connect directly the various quantities with σ_{mn}.

Motivated by the known properties of the Wigner and Weyl functions, we prove analogous properties for the extended Wigner and extended Weyl functions. First, we prove that

$$\frac{1}{4\pi^2}\iint W_e(x, p, X, P)\,\mathrm{d}x\,\mathrm{d}p = |\tilde{W}(X, P)|^2 \tag{21}$$

$$\frac{1}{16\pi^4}\iint W_e(x, p, X, P)\,\mathrm{d}X\,\mathrm{d}P = [W(x, p)]^2 \tag{22}$$

$$\frac{1}{8\pi^3}\iiiint W_e(x, p, X, P)\,\mathrm{d}x\,\mathrm{d}p\,\mathrm{d}X\,\mathrm{d}P = \mathrm{Tr}[\rho^2]. \tag{23}$$

Equation (21) is proved through an inverse Fourier transform of equation (14), which gives

$$\frac{1}{4\pi^2}\iint \mathrm{d}x\,\mathrm{d}p\,W_e(x, p, X, P)\exp[-\mathrm{i}(\beta x - \alpha p)] = \tilde{W}(X + \tfrac{1}{2}\alpha, P + \tfrac{1}{2}\beta)\tilde{W}^*(X - \tfrac{1}{2}\alpha, P - \tfrac{1}{2}\beta). \tag{24}$$

In the case where $\alpha = \beta = 0$ equation (24) gives equation (21). In a similar way we prove equation (22). Equation (23) can be proved through integration of either equation (21) or (22), using equation (10).

Equations (21), (22) in conjuction with equations (17)–(20) from [7] lead to the result

$$\frac{1}{8\pi^3}\iiint W_e(x, p, X, P)\,\mathrm{d}X\,\mathrm{d}P\,\mathrm{d}p = \int |\langle x + \tfrac{1}{2}x'|\rho|x - \tfrac{1}{2}x'\rangle|^2\mathrm{d}x' \tag{25}$$

$$\frac{1}{8\pi^3}\iiint W_e(x,p,X,P)\,\mathrm{d}X\,\mathrm{d}P\,\mathrm{d}x = \int |\langle p+\tfrac{1}{2}p'|\rho|p-\tfrac{1}{2}p'\rangle|^2\,\mathrm{d}p' \quad (26)$$

$$\frac{1}{8\pi^3}\iiint W_e(x,p,X,P)\,\mathrm{d}x\,\mathrm{d}p\,\mathrm{d}X = \int |\langle p|\rho|p+P\rangle|^2\,\mathrm{d}p \quad (27)$$

$$\frac{1}{8\pi^3}\iiint W_e(x,p,X,P)\,\mathrm{d}x\,\mathrm{d}p\,\mathrm{d}P = \int |\langle x|\rho|x+X\rangle|^2\,\mathrm{d}x \quad (28)$$

For pure states $|s\rangle$ the above equations give

$$\frac{1}{8\pi^3}\iiint W_e(x,p,X,P)\,\mathrm{d}X\,\mathrm{d}P\,\mathrm{d}p = \int |s(x+\tfrac{1}{2}x')|^2|s(x-\tfrac{1}{2}x')|^2\,\mathrm{d}x' \quad (29)$$

$$\frac{1}{8\pi^3}\iiint W_e(x,p,X,P)\mathrm{d}X\,\mathrm{d}P\,\mathrm{d}x = \int |\tilde{s}(p+\tfrac{1}{2}p')|^2|\tilde{s}(p-\tfrac{1}{2}p')|^2\,\mathrm{d}p' \quad (30)$$

$$\frac{1}{8\pi^3}\iiint W_e(x,p,X,P)\,\mathrm{d}x\,\mathrm{d}p\,\mathrm{d}X = \int |\tilde{s}(p)|^2|\tilde{s}(p+P)|^2\,\mathrm{d}p \quad (31)$$

$$\frac{1}{8\pi^3}\iiint W_e(x,p,X,P)\,\mathrm{d}x\,\mathrm{d}p\,\mathrm{d}P = \int |s(x)|^2|s(x+X)|^2\,\mathrm{d}x \quad (32)$$

where $s(x) \equiv \langle x|s\rangle$ and $\tilde{s}(p) \equiv \langle p|s\rangle$ are the wavefunctions of the state $|s\rangle$ in the x- and p-representations, correspondingly. Note that convolutions of the probability functions $|s(x)|^2$ and $|\tilde{s}(p)|^2$ appear in equations (29), (30), correspondingly; and correlations of these probability functions appear in equations (32), (31), correspondingly. The integral of the extended Wigner function with respect to X, P, p gives the convolution of the probability distribution $|s(x)|^2$; the integral with respect to X, P, x gives the convolution of the probability distribution $|\tilde{s}(p)|^2$; the integral with respect to x, p, X gives the correlation of the probability distribution $|\tilde{s}(p)|^2$; and the integral with respect to x, p, P gives the correlation of the probability distribution $|s(x)|^2$. These equations lead to the interpretation of the extended Wigner function as a pseudo-probability distribution for the convolutions and correlations of the functions $|s(x)|^2$ and $|\tilde{s}(p)|^2$.

We also prove the properties

$$\frac{1}{4\pi^2}\iint \tilde{W}_e(x',p',X',P')\,\mathrm{d}x'\,\mathrm{d}p' = [\tilde{W}^*(\tfrac{1}{2}X',\tfrac{1}{2}P')]^2 \quad (33)$$

$$\frac{1}{16\pi^4}\iint \tilde{W}_e(x',p',X',P')\,\mathrm{d}X'\,\mathrm{d}P' = W(\tfrac{1}{2}x',\tfrac{1}{2}p')W(-\tfrac{1}{2}x',-\tfrac{1}{2}p'). \quad (34)$$

Equation (33) is proved through an inverse Fourier transform of equation (15) which gives

$$\frac{1}{4\pi^2}\iint \mathrm{d}x'\,\mathrm{d}p'\,\tilde{W}_e(x',p',X',P')\exp[-\mathrm{i}(\alpha p'-\beta x')]$$
$$= \tilde{W}^*(\alpha+\tfrac{1}{2}X',\beta+\tfrac{1}{2}P')\tilde{W}(\alpha-\tfrac{1}{2}X',\beta-\tfrac{1}{2}P'). \quad (35)$$

In the case where $\alpha = \beta = 0$, equation (35) gives equation (33). In a similar way we prove equation (34).

Finally, we prove the following properties that involve squares of the Wigner and Weyl functions:

$$\frac{1}{(2\pi)^2}\iint W_e^2(x,p,X,P)\,\mathrm{d}x\,\mathrm{d}p$$
$$= \iint |\tilde{W}(X-\tfrac{1}{2}X',P-\tfrac{1}{2}P')|^2|\tilde{W}(X+\tfrac{1}{2}X',P+\tfrac{1}{2}P')|^2\,\mathrm{d}X'\,\mathrm{d}P' \quad (36)$$

$$\frac{1}{(2\pi)^6}\iint W_e^2(x,p,X,P)\,\mathrm{d}X\,\mathrm{d}P$$

$$= \iint [W(x + \tfrac{1}{2}x', p + \tfrac{1}{2}p')]^2 [W(x - \tfrac{1}{2}x', p - \tfrac{1}{2}p')]^2 \, \mathrm{d}x' \, \mathrm{d}p' \tag{37}$$

$$\frac{1}{(2\pi)^2} \iint |\tilde{W}_e(x', p', X', P')|^2 \, \mathrm{d}x' \, \mathrm{d}p'$$
$$= \iint |\tilde{W}(X + \tfrac{1}{2}X', P + \tfrac{1}{2}P')|^2 |\tilde{W}(X - \tfrac{1}{2}X', P - \tfrac{1}{2}P')|^2 \, \mathrm{d}X \, \mathrm{d}P \tag{38}$$

$$\frac{1}{(2\pi)^6} \iint |\tilde{W}_e(x', p', X', P')|^2 \, \mathrm{d}X' \, \mathrm{d}P'$$
$$= \iint [W(x + \tfrac{1}{2}x', p + \tfrac{1}{2}p')]^2 [W(x - \tfrac{1}{2}x', p - \tfrac{1}{2}p']^2 \, \mathrm{d}x \, \mathrm{d}p. \tag{39}$$

In order to prove these equations we multiply equations (14), (15) with themselves to obtain delta functions which after integration give the results appearing in the right-hand side.

4. Examples

As a first example we consider the coherent states $|A\rangle$ of equation (1) for which the Wigner and the Weyl functions are

$$W(A; x, p) = \frac{1}{\pi} \exp[-(x - \sqrt{2}A_R)^2 - (p - \sqrt{2}A_I)^2] \tag{40}$$

$$\tilde{W}(A; X, P) = \exp[-\tfrac{1}{4}(X^2 + P^2) + \mathrm{i}\sqrt{2}(XA_I - PA_R)] \tag{41}$$

where A_R, A_I are the real and imaginary parts of A. Using them we find that the extended Wigner and extended Weyl functions are

$$W_e(A; x, p, X, P) = 8\pi \exp[-2(x - \sqrt{2}A_R)^2 - 2(p - \sqrt{2}A_I)^2 - \tfrac{1}{2}P^2 - \tfrac{1}{2}X^2] \tag{42}$$

$$\tilde{W}_e(A; x', p', X', P') = 2\pi \exp[-\tfrac{1}{2}(x'^2 + p'^2) - \tfrac{1}{8}(X'^2 + P'^2)] \exp[\mathrm{i}\sqrt{2}(A_R P' - A_I X')]. \tag{43}$$

The extended Wigner function is a Gaussian located around the point $(\sqrt{2}A_R, \sqrt{2}A_I, 0, 0)$. The last two zeros reflect the fact that the Weyl function of any state is always located around $(X = 0, P = 0)$ as explained in [5]. The extended Weyl function is also Gaussian and contains the oscillatory term $\exp[\mathrm{i}\sqrt{2}(A_R P' - A_I X')]$.

As a second example we consider the thermal density matrix

$$\rho = (1 - \mathrm{e}^{-\beta}) \sum_{N=0}^{\infty} (\mathrm{e}^{-\beta N}) |N\rangle\langle N| \tag{44}$$

for which the Wigner and the Weyl functions are

$$W(x, p) = \frac{1}{\pi} \tanh(\tfrac{1}{2}\beta) \exp\{-[\tanh(\tfrac{1}{2}\beta)](x^2 + p^2)\} \tag{45}$$

$$\tilde{W}(X, P) = \exp\{-\tfrac{1}{4}[\coth(\tfrac{1}{2}\beta)](X^2 + P^2)\}. \tag{46}$$

Using them we find that the extended Wigner function is

$$W_e(x, p, X, P) = 8\pi[\tanh(\tfrac{1}{2}\beta)] \exp\{-2\tanh(\tfrac{1}{2}\beta)(x^2 + p^2) - [2\tanh(\tfrac{1}{2}\beta)]^{-1}(X^2 + P^2)\}. \tag{47}$$

This is a special case of a Gaussian function of four variables. We note that for the 'squeezed thermal density matrix' (which is the thermal density matrix multiplied on both sides with the squeezing operator) the extended Wigner function will have exponential xp and XP terms;

but it will *not* have xX, xP, pX, pP terms because they are inconsistent with the property $W_e(x, p, X, P) = W_e(x, p, -X, -P)$ which we proved earlier. The extended Weyl function for the thermal density matrix is

$$\tilde{W}_e(x', p', X', P') = 2\pi[\tanh(\tfrac{1}{2}\beta)]\exp\{-\tfrac{1}{2}\tanh(\tfrac{1}{2}\beta)(x'^2 + p'^2) - [8\tanh(\tfrac{1}{2}\beta)]^{-1}(X'^2 + P'^2)\}. \quad (48)$$

In the limit of $\beta \to \infty$ (zero temperature) the thermal density matrix becomes the pure state $|0\rangle\langle 0|$. In this case the above result agrees with equations (42), (43) with $A = 0$.

5. Quantum tomography for extended Wigner functions

The quantum tomography method of reconstructing the Wigner function from optical homodyne measurements can also give the extended Wigner and extended Weyl functions. The quantity measured is the Radon transform [10] of the Wigner function along the line $x\sin\theta - p\cos\theta = q$, defined as

$$Q(q,\theta) = \iint W(x,p)\delta(x\sin\theta - p\cos\theta - q)\,\mathrm{d}x\,\mathrm{d}p = \int W(q\sin\theta + u\cos\theta, -q\cos\theta + u\sin\theta)\,\mathrm{d}u \quad (49)$$

where q is a real variable and $-\frac{\pi}{2} \leqslant \theta < \frac{\pi}{2}$. From the $Q(q,\theta)$ we can evaluate the Weyl function using the Fourier transform

$$\tilde{W}(X,P) = \int Q\left[q, \tan^{-1}\left(\frac{P}{X}\right)\right]\exp[-\mathrm{i}q(X^2 + P^2)^{1/2}]\,\mathrm{d}q. \quad (50)$$

This formula can be easily proved by taking the Fourier transform of both sides in equation (49). Inserting equation (50) into (14) we get

$$W_e(x,p,X,P) = \iiiint \mathrm{d}X'\,\mathrm{d}P'\,\mathrm{d}q_1\,\mathrm{d}q_2\, Q\left[q_1, \tan^{-1}\left(\frac{P+\frac{1}{2}P'}{X+\frac{1}{2}X'}\right)\right] \times Q\left[q_2, \tan^{-1}\left(\frac{P-\frac{1}{2}P'}{X-\frac{1}{2}X'}\right)\right]\exp[\mathrm{i}(X'p - P'x)]\exp\{\mathrm{i}q_1[(X+\tfrac{1}{2}X')^2 + (P+\tfrac{1}{2}P')^2]^{1/2}\}\exp\{-\mathrm{i}q_2[(X-\tfrac{1}{2}X')^2 + (P-\tfrac{1}{2}P')^2]^{1/2}\}. \quad (51)$$

A similar expression can be derived for the extended Weyl function in terms of $Q(q,\theta)$.

As an example, we consider the mixed state

$$\rho = \tfrac{1}{2}(|A\rangle\langle A| + |B\rangle\langle B|) \quad (52)$$

with $A = A_R + \mathrm{i}A_I$ and $B = B_R + \mathrm{i}B_I$. The Radon transform of the Wigner function $Q(q,\theta)$ for this state has been calculated using equation (49) and is shown in figure 1.

The Wigner and Weyl functions can be easily expressed in terms of the Wigner and Weyl functions for coherent states given in equations (40), (41):

$$W(x,p) = \tfrac{1}{2}[W(A;x,p) + W(B;x,p)] \quad (53)$$

$$\tilde{W}(X,P) = \tfrac{1}{2}[\tilde{W}(A;X,P) + \tilde{W}(B;X,P)]. \quad (54)$$

Using them we calculate the extended Wigner function:

$$W_e(x,p,X,P) = \tfrac{1}{4}W_e(A;x,p,X,P) + \tfrac{1}{4}W_e(B;x,p,X,P) + \tfrac{1}{2}[W_e(A;x,p,X,P)W_e(B;x,p,X,P)]^{1/2}\exp[(A_R - B_R)^2 + (A_I - B_I)^2]\cos[\sqrt{2}X(A_I - B_I) - \sqrt{2}P(A_R - B_R)]. \quad (55)$$

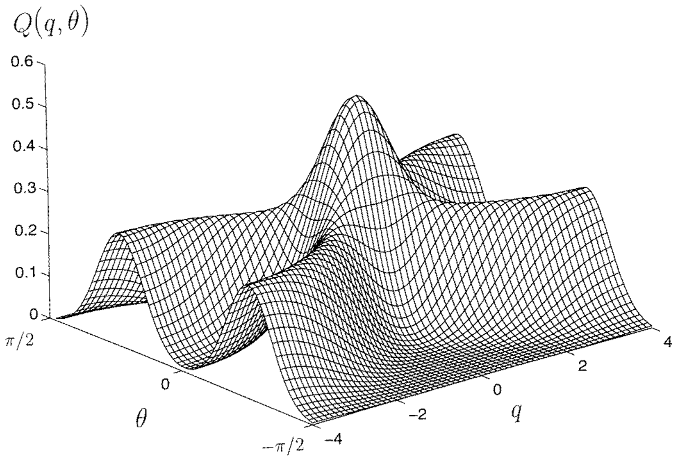

Figure 1. The Radon transform $Q(q, \theta)$ of the Wigner function for the mixed state of equation (52) with $A = -B = 2$.

It can be seen that it contains two 'auto terms' $W_e\ (A; x, p, X, P)$ and $W_e\ (B; x, p, X, P)$ which are simply the extended Wigner functions for the coherent states $|A\rangle$ and $|B\rangle$, given in equation (42). The third 'oscillatory term' appears in spite of the fact that the state is mixed. From a mathematical point of view this is related to the quartic nonlinearity of the extended Wigner function. Physically, it shows that there exists higher-order interference in the mixed state of equation (52). In figure 2 we plot the extended Wigner function for the mixed state of equation (52) with $A = -B = 2$. The results are plotted as a function of x, p (with $X = P = 0$). The two side peaks correspond to the two first auto terms in equation (55). The middle peak corresponds to the third oscillatory term in equation (55) (although the oscillations are not seen because X, P are fixed to zero). The middle peak quantifies *higher-order interference* in the *mixed* state of equation (52). The fact that the state is mixed implies the absence of interference in the Wigner and Weyl functions which are *quadratic* functions of the wavefunction. However, interference still persists in quantities which are higher-order functions of the wavefunction: e.g., in our extended Wigner function which is a quartic function of the wavefunction. Fourth-order interference involves the product of four amplitudes which can be the product of two probabilities. Therefore, we get fourth-order interference even with mixed states.

In a similar way the extended Weyl function for the same state can be expressed as

$$\begin{aligned}\tilde{W}_e(x', p', X', P') = {} & \tfrac{1}{4}\tilde{W}_e(A; x', p', X', P') + \tfrac{1}{4}\tilde{W}_e(B; x', p', X', P') \\ & + \tfrac{1}{2}[\tilde{W}_e(A; x', p', X', P')\tilde{W}_e(B; x', p', X', P')]^{1/2}] \exp[-(A_R - B_R)^2 \\ & -(A_I - B_I)^2 \cosh[\sqrt{2}x'(A_R - B_R) + \sqrt{2}p'(A_I - B_I)]. \end{aligned} \tag{56}$$

It can be seen that it contains two auto terms $\tilde{W}_e\ (A; x', p', X', P')$ and $\tilde{W}_e\ (B; x', p', X', P')$, which are the extended Weyl function for the coherent states $|A\rangle$ and $|B\rangle$ given in equation (43). The third term is related to the quartic nonlinearity of the extended Weyl function and indicates the existence of higher-order correlations in the mixed state of equation (52). In figure 3 we plot the extended Weyl function for the mixed state of equation (52) with $A = -B = 2$. The results are plotted as a function of x', p' (with $X' = P' = 0$). The middle peak corresponds to the first two auto terms in equation (56).

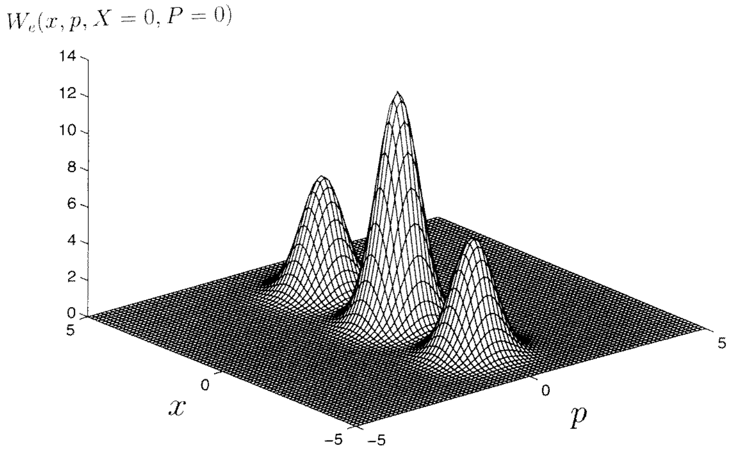

Figure 2. The extended Wigner function as a function of x, p (with $X = P = 0$) for the mixed state of equation (52) with $A = -B = 2$. The two side peaks correspond to the first two auto terms in equation (55). The middle peak corresponds to the third term in equation (55).

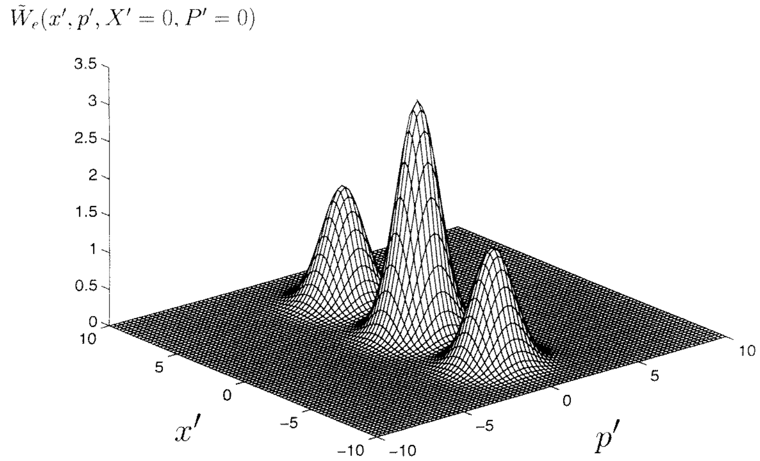

Figure 3. The extended Weyl function as a function of x', p' (with $X' = P' = 0$) for the mixed state of equation (52) with $A = -B = 2$. The middle peak corresponds to the first two auto terms in equation (56). The two side peaks correspond to the third term in equation (56).

The two side peaks correspond to the third term which demonstrates the existence of *higher-order correlations* in the *mixed* state of equation (52). The fact that the state is mixed implies the absence of correlation in quantities which are *quadratic* functions of the wavefunction. However, correlations still persist in quantities which are higher-order functions of the wavefunction.

Equations (55), (56) give the extended Wigner and extended Weyl functions for the *mixed* state of equation (52). If the corresponding pure state

$$|s\rangle = \mathcal{N}(|A\rangle + |B\rangle) \qquad \mathcal{N} = \{2 + 2\exp[-\tfrac{1}{2}|A - B|^2]\cos(A_R B_I - A_I B_R)\}^{-1/2} \tag{57}$$

is considered then there are several extra interference terms. Figure 4 shows the Radon transform of the Wigner function $Q(q, \theta)$ for this state with $A = -B = 2$ (it has been

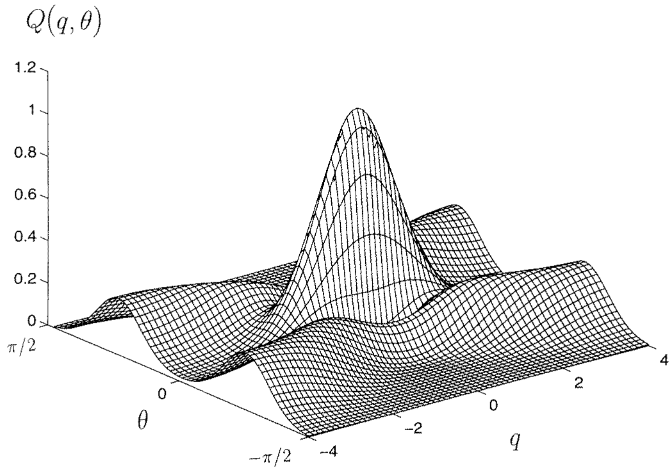

Figure 4. The Radon transform $Q(q, \theta)$ of the Wigner function for the pure state of equation (57) with $A = -B = 2$.

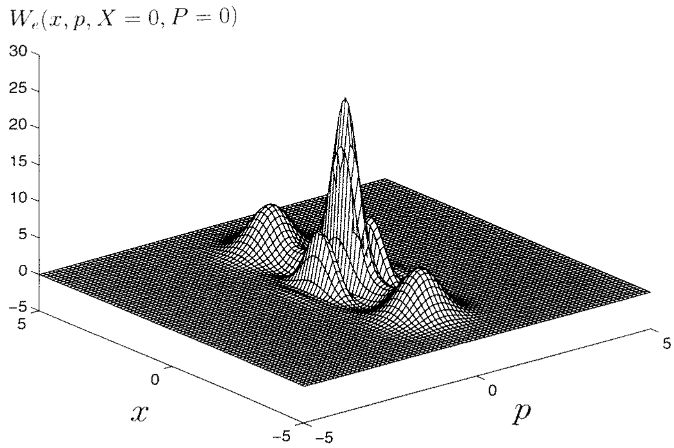

Figure 5. The extended Wigner function as a function of x, p (with $X = P = 0$) for the pure state of equation (57) with $A = -B = 2$.

calculated using equation (49)). Figure 5 shows the corresponding extended Wigner function. The results are plotted as a function of x, p (with $X = P = 0$). Comparison of figures 2 and 5 shows that the two side peaks are almost the same (note the different scales in the two diagrams); and that there is a significant increase in the middle peak in figure 5, due to the extra interference terms.

6. Discussion

The Wigner function has been studied extensively because it clearly shows quantum interference phenomena (for a review see [11]). In [5] we have studied the Weyl function and shown that it demonstrates clearly the correlations in quantum states. Both the Wigner and Weyl functions are quadratic functions of the wavefunction and are related to each through a two-dimensional Fourier transform. In this paper we have introduced the extended Wigner and extended Weyl functions which are quartic functions of the wavefunction, and show, simultaneously, both quantum noise (described by $\delta x, \delta p$) and quantum correlations (described by $\delta X, \delta P$). Many properties of these functions have been proven.

The extended Wigner and the extended Weyl functions for coherent states, thermal states and the mixed state of equation (52), have been explicitly calculated. Of particular interest is the last example which has clearly demonstrated the existence of higher-order interference in mixed states. This is because fourth-order interference involves the product of four amplitudes which can be the product of two probabilities and this is non-zero for mixed states. Of course, it can be the product of the first amplitude times the second amplitude in the third power; and that is why we get much stronger fourth-order interference with the pure state of equation (57) (as comparison of figures 2 and 5 shows).

The construction of the extended Wigner function from quantum tomography experiments has been discussed (equations (49)–(51)). The general method has been applied to the examples of superposition and also to a mixture of two coherent states. The spirit of the paper is to motivate further work on 'extended Wigner tomography' with the aim of showing explicitly the deeper connection between quantum noise and quantum correlations.

The extended Wigner function $W_e(x, p, X, P)$ and the extended Weyl function $\tilde{W}_e(x', p', X', P')$ are related through a four-dimensional Fourier transform (equation (16)). We can, therefore, continue with the same philosophy and introduce an eight-dimensional phase space x–p–X–P–x'–p'–X'–P' and corresponding Wigner and Weyl functions. More generally, we can introduce a 2^N-dimensional phase space and corresponding Wigner and Weyl functions.

We believe that the extended phase space formalism is the appropriate tool for the study of correlations and noise in quantum phenomena.

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References

[1] Balazs N L and Jennings B K 1984 *Phys. Rep.* **104** 347
Hillery M, O'Connell R F, Scully M O and Wigner E P 1984 *Phys. Rep.* **106** 121
Cahill K E and Glauber R J 1969 *Phys. Rev.* **177** 1857
Cahill K E and Glauber R J 1969 *Phys. Rev.* **177** 1882
Kim Y S and Noz M E 1986 *Theory and Applications of the Poincaré Group* (Dordrecht: Reidel)
[2] Moyal J E 1949 *Proc. Camb. Phil. Soc.* **45** 99
Bartlett M S and Moyal J E 1949 *Proc. Camb. Phil. Soc.* **45** 545
Berezin F A 1974 *Math. USSR Izv.* **8** 1109
Berezin F A 1975 *Math. USSR Izv.* **9** 341
Berezin F A 1975 *Commun. Math. Phys.* **40** 153
[3] Grossmann A 1976 *Commun. Math. Phys.* **48** 191
Royer A 1977 *Phys. Rev.* A **15** 449
Royer A 1991 *Phys. Rev.* A **43** 44
Royer A 1992 *Phys. Rev.* A **45** 793
Englert B G 1989 *J. Phys. A: Math. Gen.* **22** 625
Bishop R F and Vourdas A 1994 *Phys. Rev.* A **50** 4488
[4] Vogel K and Risken H 1989 *Phys. Rev.* A **40** 2847

Smithey D T, Beck M, Raymer M G and Faridani T 1993 *Phys. Rev. Lett.* **70** 1244
Leibfried D *et al* 1996 *Phys. Rev. Lett.* **77** 4281
Kurtsiefer C, Pfau T and Mlynek J 1997 *Nature* **386** 150
Breitenbach G, Schiller S and Mlynek J 1997 *Nature* **387** 471
[5] Chountasis S and Vourdas A 1998 *Phys. Rev.* A **58** 848
[6] Vourdas A and Sanders B 1998 *Europhys. Lett.* **43** 659
[7] Chountasis S and Vourdas A 1998 *Phys. Rev.* A **58** 1794
[8] Narcowich F J and O'Connell R F 1986 *Phys. Rev.* A **1** 34
[9] Dodonov V V and Man'ko V I 1989 *Invariants and the Evolution of Non-stationary Quantum Systems* ed M A Markov (New York: Novak)
[10] Gelfand I M, Graev M I and Vilenkin N Ya 1966 *Generalized Functions* vol 5 (London: Academic)
Ludwig D 1966 *Commun. Pure Appl. Math.* **19** 49
[11] Buzek V and Knight P L 1995 *Prog. Opt.* **34** 1